Linear and non-linear refractive indices in Riemannian and topological spaces

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The refractive index and curved space relation is formulated using the Riemann-Christoffel curvature tensor. As a consequence of the fourth rank tensor of the Riemann-Christoffel curvature tensor, we found that the refractive index should be a second rank tensor. The second rank tensor of the refractive index describes a linear optics. It implies naturally that the Riemann-Christoffel curvature tensor is related to the linear optics. In case of a non-linear optics, the refractive index is a sixth rank tensor, if susceptibility is a fourth rank tensor. The Riemann-Christoffel curvature tensor can be formulated in the non-linear optics but there exist a gradient of susceptibility term. In topological space, we see that the linear and non-linear refractive indices are related to the Euler-Poincare characteristic, $\chi(M)$. Because the Euler-Poincare characteristic is topological invariant then the linear and non-linear refractive indices are also topological invariants.

I. INTRODUCTION

What is really happened if light passes through a medium? This question becomes more interesting nowadays related to conceptual development and technological innovation. One of the very important idea to understand this question is the refractive index or the index of refraction, i.e. measure of the bending of a ray of light when passing from one medium into another[1]. The refractive index of a medium is an optical parameter, since it exhibits the optical properties of the material[2].

The refractive index, n, is defined as velocity of light of a given wavelength in empty space or vacuum (c) divided by its velocity in a substance, v,[3]

$$n = \frac{c}{v} \tag{1}$$

It[1] describes how matter affects light propagation, through the electric permittivity, ε, and the magnetic permeability, μ[4]

$$n = \sqrt{\left(\frac{\varepsilon}{\varepsilon_0}\right)\left(\frac{\mu}{\mu_0}\right)} = \sqrt{(\varepsilon_r)(\mu_r)} \tag{2}$$

where ε_0 and μ_0 are the permittivity and the permeability of vacuum respectively, ε_r and μ_r is relative permittivity and relative permeability of non-vacuum medium respectively, which the values are relative i.e. they depend on the characteristics of medium[4,5].

In the most substrates, the refractive index decreases by increasing temperature[3]. A denser material generally tends to have a larger refraction index[6]. The refractive index in an fibre optic can be changed due to external forces such as the tensile force, the bending force[7].

Mathematically, the refractive index is a zeroth rank tensor (scalar) and it can not be a first rank tensor (vector), but it can be a second rank tensor, a third rank tensor or a higher rank tensor (which is well known as non-linear phenomena of second order, third order, etc)[8]. The refractive index is the zeroth rank tensor, if the medium or material is isotropic[2]. Generally, the refractive index is written as the second rank tensor, n_{ij}, a 3×3 matrix, if the material is linear[3]. It can be the third rank tensor or the fourth rank tensor if the material is non-linear[11].

The refractive index has a large number of applications. It is mostly applied to identify a particular substance, to confirm its purity or to measure its concentration. It also can be used in determination of drug concentration in pharmaceutical industry, to calculate a focusing power of lenses and a dispersive power of prisms. Also, it can be applied to estimate a thermophysical properties of hydrocarbons and petroleum mixtures[3].

II. THE LINEAR REFRACTIVE INDEX IN THE RIEMANNIAN SPACE

Let $f(r)$ be defined and differentiable at a point (r) in a certain region of space (i.e. f defines a differentiable scalar field). Then the gradient of f in the spherical coordinate is defined by

$$\vec{\nabla} f \equiv \frac{df}{dr}\ \hat{r} \tag{3}$$

In the tensor form[12]

$$\vec{\nabla} f = \text{grad } f = f_{,j} = \frac{\partial f}{\partial x^j} \tag{4}$$

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[1] The sign of the refractive index is often taken as positive, but in 1968 Veselago shows that there are substrates with negative permitivity and negative permeability. In these substrates, the refractive index has a negative value[3].

[2] Isotropy comes from the Greek words *isos* (equal) and *tropos* (way): uniform in all directions[10]. An isotropic material is a material that has the same optical properties, regardless of the direction in which light propagates through the material[9,10].

[3] Linear material is a material that when exposed to light at a certain frequency will generate light with the same frequency[5].

where $f_{,j}$ is the covariant derivative of f with respect to x^j. Here, $\vec{\nabla} f$ defines a vector field i.e. the gradient of a scalar field is a vector field[12,13].

Let us analyse the equation below[14–16]

$$\frac{1}{R} = \hat{N} \ . \ \vec{\nabla} \ \ln \ n(r) \tag{5}$$

where R is a radius of curvature, $\hat{N}$ is an unit vector along the principal normal or has the same direction with $\vec{\nabla}$ ln $n(r)$ and $n(r)$ is the space dependent refractive index. Eq.(5) tells us that the rays are therefore bent in the direction of increasing refractive index[14]. The illustration is given in **Figure 1** below.

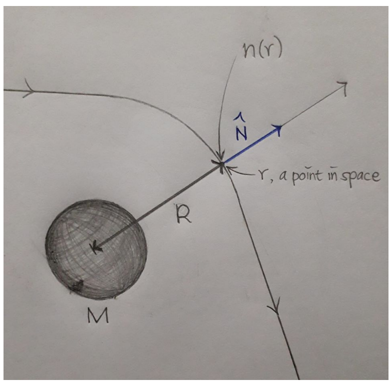

Fig. 1 The illustration of the refractive index as a function of curvature.

Let us define that

$$\hat{N} \equiv \frac{\vec{\nabla} n(r)}{|\vec{\nabla} n(r)|} \tag{6}$$

Using notation of gradient operator in (3) and substituting (6) into (5), we obtain

$$\begin{aligned} \frac{1}{R} &= \frac{\vec{\nabla} n(r)}{|\vec{\nabla} n(r)|} \ . \ \vec{\nabla} \ \ln \ n(r) \\ &= \frac{\frac{dn(r)}{dr} \ \hat{r}}{|\frac{dn(r)}{dr} \ \hat{r}|} \ . \ \frac{d}{dr} \ \hat{r} \int \frac{1}{n(r)} \ dn(r) \end{aligned} \tag{7}$$

Because $\hat{r}.\hat{r} = |\hat{r}||\hat{r}| \cos 0^0 = 1$, $\hat{r}$ is a unit vector which its magnitude i.e. $|\hat{r}|$ is 1, then eq.(7) becomes

$$\frac{1}{R} = \frac{1}{n(r)} \ \left(\frac{dn(r)}{dr}\right)^2 \ \left(\left|\frac{dn(r)}{dr}\right|\right)^{-1} \tag{8}$$

If we assume that the derivative of a function $n(r)$ always takes a positive value then

$$\left|\frac{dn(r)}{dr}\right| = \frac{dn(r)}{dr} \tag{9}$$

So, eq.(8) becomes

$$\frac{1}{R} = \frac{1}{n(r)} \ \frac{dn(r)}{dr} \tag{10}$$

where $n(r)$ can be e.g. exponential, logarithmic, quadratic, linear functions.

What is precisely the form of function $n(r)$? We assume that the curvature of space is produced by a spherically symmetric body with mass at rest. So, the mass of this body produces the static spherically symmetric gravitational field. The static condition means that with a static coordinate system, the metric tensors, $g_{\mu\nu}$, are independent of time, x^0 or t, and also $g_{tm} = 0$, for $m = r, \ \theta, \ \phi^{?}$.

In tensor form, by using relation in eq.(4) and tensor notation N_k, i.e. the first rank tensor for $\hat{N}$, the right hand side of eq.(5) can be written as

$$N_k \ \frac{\partial}{\partial x^j} [\ln \ n(r)] \tag{11}$$

Because of the Riemann-Christoffel curvature tensor is the fourth rank tensor, so *the refractive index in eq.(11) should be written as the second rank tensor*. We obtain the relation between the curvature tensor and the refractive index tensor as below[17]

$$\frac{R_{mijk}}{g} = N_k \ \frac{\partial \ln \ n_{mi}}{\partial x^j} \tag{12}$$

Eq.(12) implies that the curved space which is described by the Riemann-Christoffel curvature tensor related naturally to linear optics.

III. THE NON-LINEAR REFRACTIVE INDEX IN THE RIEMANNIAN SPACE

How about the form of the non-linear refractive index i.e. the refractive index related to the non-linear optics? In optics, *non-linear properties of materials are usually described by non-linear susceptibilities*[18]. Mathematically, the optical non-linear response can be expressed as a relationship between the polarization density[4], $\vec{P}$, and the electric field, $\vec{E}$.

In the linear case, a relation between the polarization density and the electric field is simply expressed as[20,21]

$$\vec{P} = \varepsilon_0 \ \chi^{(1)} \ \vec{E} \tag{13}$$

where ε_0 is the permittivity of vacuum space, $\chi^{(1)}$ is the first order susceptibility or linear susceptibility and it is a scalar, whereas the polarization and the electric field are vectors.

In the non-linear case[5], the polarization density can be modelled as a power series of the electric field, $\vec{E}$, as

[4] Light is an electromagnetic wave, and the electric field of this wave oscillates perpendicularly to the direction of light propagation. If the direction of the electric field of light is well defined, it is polarized light. The most common source of polarized light is a laser[19].

[5] A non-linear system is a system in which the change of the output is not proportional to the change of the input[22]. In optics, the non-linearity is typically observed only at very high intensities (field strength) of light such as those provided by lasers[23].

below[20,21,24]

$$\vec{P} = \varepsilon_0 \left[\chi^{(1)}\ \vec{E} + \chi^{(2)}\ \vec{E}^2 + \chi^{(3)}\ \vec{E}^3 + ...\right] \\ = \vec{P}^1 + \vec{P}^2 + \vec{P}^3 + ... \quad (14)$$

where $\vec{E}^2 = \vec{E}\ \vec{E}$, $\vec{E}^3 = \vec{E}\ \vec{E}\ \vec{E}$ etc. $\vec{P}^1$ is called the linear polarization while $\vec{P}^2$, $\vec{P}^3$, are called the second and third non-linear polarizations, respectively. Thus, *the polarization is composed by linear and non-linear components*[21]. The first susceptibility term, $\chi^{(1)}$, corresponds to the linear susceptibility (dimensionless). The subsequent non-linear susceptibilities, $\chi^{(a)}$, where $a > 1$, have units of $(\text{meter/volt})^{a-1}$[25,26]. The quantities $\chi^{(2)}$ and $\chi^{(3)}$ are known as the second order and third order susceptibilities, respectively. These electric susceptibilities, $\chi^{(1)}$, $\chi^{(2)}$, $\chi^{(3)}$, are the second, third and fourth rank tensors, respectively[20]. *In general, the non-linear susceptibilities depend on the frequencies of the applied fields*[27]. In optical Kerr effect, the third order susceptibility, $\chi^{(3)}$, related to the non-linear refractive index[9].

Now, we have a question: if the non-linear refractive index is related to the third order susceptibility, $\chi^{(3)}$, and the third order susceptibility is the fourth rank tensor[20] then how to define the non-linear refractive index which is related to the fourth rank tensor of the third order susceptibility?

For linearly polarized monochromatic light in an isotropic medium or a cubic crystal, the non-linear refractive index, n_2, can be expressed by[24]

$$n_0 = 12\pi\ (n_2)^{-1}\ \text{Re}\ \chi^{(3)} \quad (15)$$

where n_0 is a linear refractive index and Re $\chi^{(3)}$ is a real part[6] of the third order non-linear susceptibility. We see from eq.(15), the linear refractive index is a function of the non-linear refractive index. Some values of the linear refractive index, n_0, and the non-linear refractive index, n_2, of some oxides are shown in Table I of Dimitrov-Sakka[24]. Figure 1 of Dimitrov-Sakka shows the "exponential relation" between the linear refractive index, n_0, and the non-linear refractive index, n_2[24]. It means that the non-linear refractive index increases exponentially with the increasing of the linear refractive index.

We see from eq.(12), the linear refractive index, n_0, is the second rank tensor, n_{mi}, and refer to Jatirian, et al.[20] the (real) third order susceptibility, $\chi^{(3)}$, is the fourth rank tensor, $\chi^{(3)}_{pqrs}$, so we can write eq.(15) as below

$$n_{mi} = 12\pi\ n^{pqrs}_{mi}\ \chi^{(3)}_{pqrs} \quad (16)$$

where $n^{pqrs}_{mi} = (n_2)^{-1}$. So,

$$n_2 = n^{mi}_{pqrs} \quad (17)$$

[6] In general, susceptibility is a complex quantity. The real part is related to the refraction, while the imaginary part is related to the absorption.

It means that *the non-linear refractive index should be the sixth rank tensor (a mixed tensor of second rank cotravariant and fourth rank covariant)*.

Substituting (16) into (12), we obtain

$$\frac{R_{mijk}}{g} = N_k\ \frac{\partial}{\partial x^j} \left\{\ln\left(12\pi\ n^{pqrs}_{mi}\ \chi^{(3)}_{pqrs}\right)\right\} \quad (18)$$

where π is a constant, so the gradient of π gives zero. Using relation $\ln\ (ABC) = \ln A + \ln B + \ln C$, then eq.(18) becomes

$$\frac{R_{mijk}}{g} - N_k\ \frac{\partial\ \ln\ \chi^{(3)}_{pqrs}}{\partial x^j} = N_k\ \frac{\partial\ \ln\ n^{pqrs}_{mi}}{\partial x^j} \quad (19)$$

Eq.(19) shows that in the non-linear optics, the Riemann-Christoffel curvature tensor is related to the sixth rank tensor of the refractive index.

IV. THE REFRACTIVE INDEX IN THE TOPOLOGICAL SPACE (GLOBAL GEOMETRY)

Riemannian geometry, which was the high dimensional generalization of Gauss intrinsic surface theory, gives a geometrical structure which is entirely local[28]. Local geometry is the study of small pieces of a manifold[29]. A manifold is a topological space that locally resembles Euclidean space near each point[30]. A topological space may be defined as a set of points, along with a set of neighbourhoods for each point, satisfying a set of axioms relating points and neighbourhoods[31]. Global geometry is, as the word suggests, the study of the total manifold including, for example, the number of holes[29].

Let Ω_{mi} be "the curvature form"

$$\Omega_{mi} = \sum\ R_{mijk}\ dx^j \wedge dx^k \quad (20)$$

where R_{mijk} be the Riemann-Christoffel tensor and x^j, x^k are local coordinates[32,33]. $\wedge$ is a notation for the exterior (wedge) product. It satisfies the distributive, anti-commutative and associative laws[33]. *Ω_{mi} is an anti-symmetric matrix of 2-forms*. In differential geometry, the curvature form describes curvature of a connection on a principal bundle. It can be considered as an alternative to or generalization of the curvature tensor in Riemannian geometry[34].

In case of the linear optics, if we substitute eq.(12) into eq.(20), we obtain

$$\Omega_{mi} = \sum\ g\left(N_k\ \frac{\partial \ln n_{mi}}{\partial x^j}\right)\ dx^j \wedge dx^k \quad (21)$$

It means that the curvature form, Ω_{mi}, is related to the second rank tensor of the linear refractive index, n_{mi}.

In case of the non-linear optics, if we substitute eq.(18) into eq.(20), we obtain

$$\Omega_{mi} = \sum\ g\left\{N_k\ \frac{\partial}{\partial x^j}\left[\ln\left(12\pi\ n^{pqrs}_{mi}\ \chi^{(3)}_{pqrs}\right)\right]\right\} \\ dx^j \wedge dx^k \quad (22)$$

It means that the curvature form, Ω_{mi}, is related to the six rank tensor of the non-linear refractive index, n_{mi}^{pqrs} and the fourth rank tensor of the third order non-linear susceptibility, $\chi_{pqrs}^{(3)}$.

For any even-dimensional complex $2n \times 2n$ antisymmetric matrix, M, we define the *pfaffian* of M, denoted by pf M, as

$$\text{pf } M = \frac{1}{2^n n!} \epsilon_{i_1 j_1 \ i_2 j_2 \ .. \ i_n j_n} \ M_{i_1 j_1} \ M_{i_2 j_2} \ .. \ M_{i_n j_n} \quad (23)$$

where ϵ is the rank-$2n$ Levi-Civita tensor and the sum over repeated indices is implied[35].

One can rewrite eq.(23) by restricting the sum over indices in such a way that removes the combinatoric factor $2^n n!$ in the denominator. Let P be the set of permutations, $\{i_1, \ i_2, \ ..., \ i_{2n}\}$ with respect to $\{1, \ 2, \ ..., \ 2n\}$, such that

$$i_1 < j_1, \ i_2 < j_2, ..., \ i_{2n} < j_{2n} \quad (24)$$

and

$$i_1 < i_2 < ... < i_{2n} \quad (25)$$

then

$$\text{pf } M = \sum_P{}' \ (-1)^P \ M_{i_1 j_1} \ M_{i_2 j_2} \ .. \ M_{i_n j_n} \quad (26)$$

where $(-1)^P = 1$ for even permutations and $(-1)^P = -1$ for odd permutations. The prime on the sum in eq.(26) has been employed to remind us that the set of permutations P is restricted according to eqs.(24), (25). *If M is an odd dimensional complex antisymmetric matrix, the corresponding pfaffian is defined to be zero*[35].

Let

$$Pf = \sum \ \epsilon_{i_1 ... i_{2n}} \ \Omega_{i_1 i_2} \wedge ... \wedge \Omega_{i_{2n-1} i_{2n}} \quad (27)$$

be the pfaffian, where $\epsilon_{i_1 ... i_{2n}}$ is +1 or -1 according as its indices form an even or odd permutation of $1, ..., 2n$, and its otherwise zero, and the sum is extended over all indices from 1 to $2n$[32].

Then the Gauss-Bonnet theorem says

$$(-1)^n \frac{1}{2^{2n} \pi^n n!} \int Pf = \chi(M) \quad (28)$$

where $\chi(M)$ is the Euler-Poincare characteristic of M. It is invariant topologically[32].

The formula

$$\int_{M^{2n}} \text{Pf} \left(\frac{1}{2\pi} \ \frac{1}{2} \ R_{mijk} \ dx^j \ dx^k \right) = \chi(M^{2n}) \quad (29)$$

where R_{mijk} is the Riemann tensor and $\chi(M^{2n})$ is the Euler characteristic, is the Gauss-Bonnet-Chern theorem for even-dimensional oriented compact Riemannian manifolds. This is a special case of the Atiyah-Singer index theorem[36].

From eq.(21)-(29), we see that the linear and non-linear refractive indices are related to the Euler-Poincare characteristic, $\chi(M)$. Because the Euler-Poincare characteristic is topological invariant then the linear and non-linear refractive indices are also topological invariants.

V. DISCUSSION AND CONCLUSION

In one and two dimensional spaces, the Gaussian curvatures are $1/R$ (a circle) and $1/R^2$ (a sphere), respectively. Because the homogeneous and isotropic spaces can be spherical[37] then the Gaussian curvature can be related with the homogeneous and isotropic spaces. This homogeneous and isotropic spaces have a constant curvature[38,39]. It means that the Gaussian curvature has a constant curvature. A sphere is an example of a surface of constant (positive) curvature. Georg Friedrich Bernhard Riemann, a student of Johann Carl Friedrich Gauss, generalize the Gauss curvature of space for more than two dimensions. The result is the Riemann-Christoffel curvature tensor where the Christoffel symbol is used in the formulation of the generalized curvature.

In relation with the refractive index, because the refractive index is related to the curvature of space for one and two dimensions, and this curvature of space can be generalized to more than two dimensions, then the refractive index should be able to be formulated in more than two dimensional curved space. It gives *the second rank tensor* of the refractive index as the consequence of the fourth rank tensor of the Riemann-Christoffel curvature tensor. The second rank tensor of the refractive index describes the linear optics. It implies that the Riemann-Christoffel curvature tensor is related naturally to the linear optics.

Because the non-linear refractive index can be expressed as a function of the linear refractive index and the third order of the susceptibility, where the linear refractive index is the second rank tensor and the third order of susceptibility is the fourth rank tensor then the non-linear refractive index should be *the sixth rank tensor*. It means that the Riemann-Christoffel curvature tensor can be related to the non-linear optics.

From eq.(21)-(28), we see that the linear and non-linear refractive indices are related to the Euler-Poincare characteristic, $\chi(M)$. Because the Euler-Poincare characteristic is topological invariant then the linear and non-linear refractive indices are also topological invariants.

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